Boundary Layer Theory.

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Asymptotic analysis:

Given $a(\varepsilon)$, we aim to understand its behavior as $\varepsilon \to 0$.

$$\frac{1}{1-\varepsilon} = 1 + \sum_{n\geq 1}^{\infty} \varepsilon^n \qquad |\varepsilon|<1.$$

$$\lim_{n\to+\infty} \left| \frac{1}{1-\varepsilon} - \sum_{k\geq 1}^{n} \varepsilon^k \right| = 0.$$

Ex: Roots of $\underline{\varepsilon x^2} + x - 1 = 0$

Ansatz: $|x| \sim 1$

So that εx^2 can be regarded as a perturbation term.

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

$$\varepsilon(1+\varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 + (1+\varepsilon x_1 + \varepsilon^2 x_2 + \cdots) - 1 = 0.$$

$O(1)$: automatically holds

$O(\varepsilon)$: $x_1 = -1$

$O(\varepsilon^2)$: $x_2 = -2x_1 = 2$

$\vdots$

$O(\varepsilon^n)$: $x_n = \dfrac{(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n)}{(n+1)!} 4^n$

Then $\hat{x} = \sum_{n=0}^{\infty} x_n \varepsilon^n$ defines a solution. $\left(|\varepsilon| < \frac{1}{4}\right)$

$\Delta = 1 + 4\varepsilon > 0$

$$\hat{x}_{\pm} = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}. \qquad \hat{x} = \hat{x}_{+}.$$

$$\hat{x}_{-} = \frac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon} \to +\infty \text{ as } \varepsilon \to 0.$$

Q: How to find $\hat{x}_-$ by asymptotic?

Rescaling: $x = \varepsilon^{-\alpha} y$. $\quad \alpha > 0$.

$$\underbrace{\varepsilon^{1-2\alpha} y^2 + \overset{\sim O(\varepsilon^{-1})}{\varepsilon^{-\alpha} y}}_{\text{balance.}} \underbrace{- 1}_{\sim \text{ lower order}} = 0$$

$\alpha = 1$

Equation of y: $\quad y^2 + y - \varepsilon = 0$

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

$O(1)$: $\quad y_0^2 + y_0 = 0$, $\quad y_0 \cancel{=} 0$ or $y_0 = -1$

$O(\varepsilon)$: $\quad y_1 = -1$

$\vdots$

$O(\varepsilon^n)$: $y_n = \dfrac{-(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} 4^{n-1}$

$\vdots$

$$\hat{x}_+ = \varepsilon^{-1} \sum_{n=0}^{\infty} \varepsilon^n y_n$$

Ex: (ODE)

$$\begin{cases} \varepsilon \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 1, \quad x \in (0,1) \\ \\ y(0) = 0, \quad y(1) = 1. \end{cases}$$

Q: Asymptotic behavior of $y^\varepsilon(x)$ as $\varepsilon \to 0^+$.

$\varepsilon = 0$. $\qquad \dfrac{dy_0}{dx} - y_0 = 1$

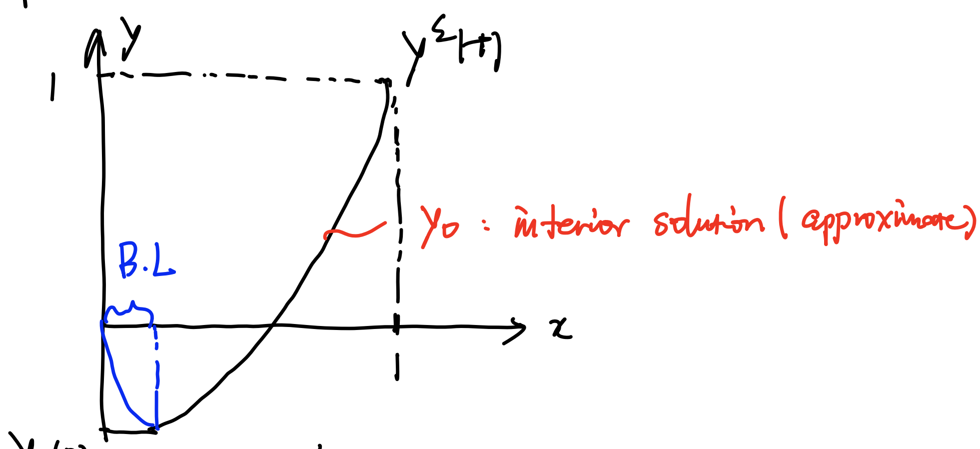
$$y_0 = Ae^x - 1$$

only one B.C is needed. $y_0(1) = 1 \Rightarrow \quad A = 2e^{-1}$

$y_0(x) = 2e^{x-1} - 1$. $\quad y_0(0) = 2e^{-1} - 1 \neq 0$!

*: y_0 approximate y^ε well if there is no boundary.

* Near the boundary $x=0$, y^ε changes dramatically from y_0 to 0.



width $\sim \varepsilon^\alpha$.

To capture the behavior near the boundary,

we introduce $z = \frac{x}{\varepsilon^\alpha}$ and assume $y^\varepsilon \approx Y(z)$

$Y \sim$ profile of B.L.

$$\underbrace{\varepsilon^{1-2\alpha} Y'' + \varepsilon^{-\alpha} Y'}_{\text{Balance}} - Y = 0$$

$\alpha = 1$, $\quad Y_0'' + Y_0' = 0$

B.C: $Y_0(0)=0, \quad \lim_{Z\to+\infty} Y(Z) = y_0(0) = 2e^{-1}-1$

$$Y_0(Z) = (1-e^{-Z})(2e^{-1}-1).$$

Approximate solution:

$$y_0(x) + Y_0(Z) - 2e^{-1}-1 = 2e^{x-1} - 1 - (2e^{-1}-1)e^{-\varepsilon^{-1}x}.$$

Proceeding as such:

$$y^{\varepsilon}(x) \sim \sum_{n=0}^{\infty}\left(y_n(x) + Y_n\left(\frac{y}{\varepsilon}\right) - y_n(0)\right)\varepsilon^n.$$

Exact solution of ODE:

$$y^{\varepsilon}(x) = \frac{e^{\lambda_1}-2}{e^{\lambda_1}-e^{\lambda_2}} e^{\lambda_2 x} + \frac{2-e^{\lambda_2}}{e^{\lambda_1}-e^{\lambda_2}} e^{\lambda_1 x} - 1$$

Here $\lambda_1 = \dfrac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$

$\lambda_2 = \dfrac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$

$\lambda_1 \sim -\varepsilon^{-1}, \quad \lambda_2 \sim 1.$

$$\Rightarrow y^{\varepsilon}(x) \sim \frac{e^{-\varepsilon^{-1}}-2}{e^{-\varepsilon^{-1}}-e} e^{x} + \frac{2-e}{e^{-\varepsilon^{-1}}-e} e^{-\varepsilon^{-1}x} - 1$$

$$\sim 2e^{x-1} - 1 - (2e^{-1} - 1)e^{-\varepsilon^{-1} x}$$

$$(e^{-\varepsilon^{-1}} << 1)$$

Q: How about solving:

$$\begin{cases} y_0' - y_0 = 1 \\ y_0(0) = 0 \end{cases}$$

in the first step?

For such case:

$$\begin{cases} y_0 = Ae^{x} - 1 \\ y_0(0) = 0 \end{cases} \Rightarrow y_0(x) = e^{x} - 1.$$

is not an interior solution!

$y_0(1) = e - 1 \neq 0.$

B.L near 1: $z = \dfrac{1-x}{\varepsilon^{\alpha}}$ $y^{\varepsilon} \sim Y(z)$ near $x = 1.$

$$\Rightarrow \varepsilon^{1-2\alpha} Y'' - \varepsilon^{-\alpha} Y' - Y = 0$$

$\alpha = 1$ $Y'' - Y' = 0,$ $Y(0) = 1,$ $Y(+\infty) = e - 1$ has no solution!

To summarize: Solving $y^{\varepsilon}(x)$:

Step 1: $d \sim 1$: Interior solution

$$y^{\varepsilon}(x) = \sum_{n=0}^{\infty} y_n(x)\,\varepsilon^n, \quad \text{regular}$$

$d \sim \varepsilon^{\alpha} \ll 1$, Boundary layer correction: $Z = \frac{x}{\varepsilon^{\alpha}}$

$$Y(Z) \sim \sum_{n=0}^{\infty} Y_n(Z)\,\varepsilon^n.$$

Step 2: Imposing boundary condition:

For each n, $y_n(0)$ solves from the equation

$Y_n(0)$ satisfies the B.C of $y^{\varepsilon}(0)$

Matching Y_n with y_n:

$$\lim_{Z\to+\infty} Y_n(Z) = y_n(0)$$

$$y^{\varepsilon}(x) \sim \sum_{n=0}^{\infty}\left(y_n(0) + Y_n\left(\frac{x}{\varepsilon^{\alpha}}\right) - y_n(0)\right)$$

Rigorous justification of expansion.

The series has no reason to convergence.

truncate + remainder estimate:

Ansatz

$$y^{\varepsilon}(x) = \underbrace{y_0(x) + Y_0\left(\frac{y}{\varepsilon}\right) - Y_0(0)}_{\text{leading order approximation}} + y_R(x)$$

and to show $\|y_R^{\varepsilon}\|_{L^{\infty}} \to 0$ as $\varepsilon \to 0^+$.

Equation of $y_R(x)$:

$$\begin{cases} \varepsilon \dfrac{d^2}{dx^2} y_R + \dfrac{d}{dx} y_R - y_R = -\varepsilon \dfrac{d^2}{dx^2} y_0(x) - \left[Y_0\left(\dfrac{y}{\varepsilon}\right) - Y_0(+\infty)\right] \\ \qquad \triangleq E(x) : \text{ - error term} \\ y_R(0) = y_R(1) = 0. \end{cases}$$

Energy estimate:

$$\int_0^1 \underbrace{-\varepsilon \frac{d^2}{dx^2} y_R \cdot y_R}_{} - \frac{d}{dx} y_R \cdot y_R + \|y_R\|_{L^2}^2 = \int_0^1 E(x)\, y_R \, dx$$

$$\varepsilon \left\|\frac{d}{dx} y_R\right\|_{L^2}^2 + \|y_R\|_{L^2}^2 \leq \|E\|_{L^2} \|y_R\|_{L^2}$$

$$\leq \frac{1}{2}\|E\|_{L^2}^2 + \frac{1}{2}\|y_R\|_{L^2}^2$$

$$\Rightarrow \quad \|y_R\|_{L^2} \leq \|E\|_{L^2}$$

$$\left\|\frac{d}{dx} y_R\right\|_{L^2} \leq \frac{1}{(2\varepsilon)^{\frac{1}{2}}} \|E\|_{L^2}$$

$\rightarrow$ negative power

$$\|y_R\|_{L^\infty} \leq \|y_R\|_{L^2}^{\frac{1}{2}} \left\|\frac{d}{dx} y_R\right\|_{L^2}^{\frac{1}{2}} \lesssim \varepsilon^{-\frac{1}{4}} \|E\|_{L^2}$$

Recall

$$\sim e^{-\frac{x}{\varepsilon}}$$

$$E = -\varepsilon \frac{d^2}{dx^2} y_0 - \left(\underline{Y_0\left(\frac{x}{\varepsilon}\right) - Y_0(+\infty)} \right)$$

$$\|E\|_{L^2} \lesssim C\varepsilon + C\varepsilon^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{2}}.$$

Therefore: $\|y_R\|_{L^\infty} \leq C\varepsilon^{\frac{1}{4}}$

and $\left\|y^\varepsilon(x) - y_0(x) - Y_0\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty} \leq o\left(\varepsilon^{\frac{1}{4}}\right) \to 0$ as $\varepsilon \to 0^+$.

<u>Remark</u>: In practice, if the remainder estimate is not so good:

e.g.: $\|y_R\|_{L^\infty} \leq \varepsilon^{-k} \|E\|_X$.

Then we need construct higher order approximate solutions:

$$y^\varepsilon(x) = \sum_{n=0}^{N} \left(y_k(x) + Y_k\left(\frac{x}{\varepsilon}\right)\right)\varepsilon^N + \varepsilon y_R(x).$$

Such that the error term: $\|E\|_X \leq \varepsilon^k$

So $\|y_R\|_{L^\infty} \leq \varepsilon^{-k}\|E\|_X \lesssim C$

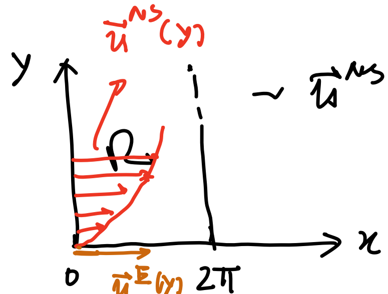
Then $\|y^\varepsilon(x) - y_0(x) - Y_0(\frac{x}{\varepsilon})\|_{L^\infty} \lesssim O(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Prandtl boundary Layer theory

Incompressible Navier-stokes equation: (Viscous fluid)

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p - \varepsilon\Delta\vec{u} = 0 \\ \nabla\cdot\vec{u} = 0 \qquad \vec{u}|_{y=0} = 0 \\ \vec{u}|_{t=0} = u_0 \end{cases}$$

— parabolic



$\sim \vec{u}^{NS} = (u^{NS}, v^{NS})$

$= 0$

$\Omega = [0, 2\pi] \times \mathbb{R}_+$

$\varepsilon = \frac{1}{Re}$: Viscosity Coefficient

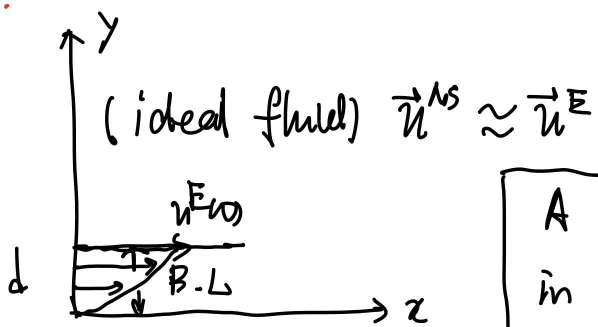
$\vec{u} = (u, v)$: velocity field, p: pressure — Hyperbolic

(Ideal fluid)

$\varepsilon \to 0^+$: $\begin{cases} \partial_t \vec{u}^E + \vec{u}^E\cdot\nabla\vec{u}^E + \nabla p^E = 0 \\ \nabla\cdot\vec{u}^E = 0 \\ \vec{u}^E|_{t=0} = u_0^E(x, y) \end{cases}$

$\vec{u}^E = (u^E, v^E)$

$\vec{u}^E \cdot \vec{n} = 0 \Rightarrow v^E|_{y=0} = 0$



A thin layer near the boundary in which the tangential velocity change rapidly from $\vec{u}^E(0)$ to 0

Q: Description of the flow near the boundary.

Ludwig Prandtl (1904)

Über Flüssigkeitsbewegung bei sehr kleiner Reibung.

On the motion of fluids in very little friction.

$d \sim \sqrt{\varepsilon}$.

$$\begin{cases} \partial_t u + u\cdot\partial_x u + v\partial_y u + \partial_x p - \varepsilon\Delta u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \varepsilon\Delta v = 0, \\ \partial_x u + \partial_y v = 0 \end{cases}$$

Rescaled variable $Y = \frac{y}{\varepsilon^\alpha}$

$(u, v) \sim (U, \varepsilon^\alpha V)(x, Y)$.

$$\begin{cases} \underbrace{\partial_t U + U\partial_x U + V\partial_Y U + \partial_x p - \varepsilon^{1-2\alpha}\partial_Y^2 U}_{\text{Balance}} - \varepsilon\partial_x^2 U = 0. \\ \partial_t V + U\partial_x V + V\partial_Y V + \varepsilon^{-2\alpha}\partial_Y p - \varepsilon^{1-2\alpha}\partial_Y^2 V - \varepsilon\partial_x^2 V = 0 \\ \partial_x u + \partial_Y v = 0 \end{cases}$$

$\alpha = \frac{1}{2}$.

Prandtl equation

$$\partial_t U + U\partial_x U + V\partial_Y U - \partial_Y^2 U = -\partial_x p^E(t.x,0)$$

$$\Rightarrow \begin{cases} \partial_t U + U\partial_x U + V\partial_Y U + \partial_x p - \partial_Y^2 U = 0 \\ \partial_Y p = 0 \qquad p(t.x.Y) \equiv p^E(t.x,0). \\ \partial_x U + \partial_Y V = 0 \end{cases}$$

Boundary condition:

$$(U, V)|_{Y=0} = (0, 0)$$

Matching condition:

$$\lim_{Y\to+\infty} U(t, x.Y) = u^E(t, x.0). \quad \lim_{Y\to+\infty} p(t.x.Y) = p^E(t.x,0)$$

Prandtl Ansatz:

$$(u^{NS}, v^{NS})(t, x, y)$$

$$\sim (u^E, v^E)(t, x, y) + \left(U^P, \sqrt{\varepsilon} V^P\right)\left(t, x, \frac{y}{\sqrt{\varepsilon}}\right) + o(1)$$

<u>Justification</u>:

Step 1: Well-posedness of Prandtl equation

Step 2: Justify the remainder tends to 0 as $\varepsilon \to 0^+$.

$$\lim_{\varepsilon \to 0^+} \sup_{0 \le t \le T_0} \| \vec{u}^{NS}(t) - \vec{u}^E(t) - \vec{u}^P(t) \|_{L^\infty(\Omega)} = 0.$$

Mathematical difficulties.

Recall the Prandtl system:

$$\begin{cases} \partial_t u^P + u^P \partial_x u^P + v^P \partial_y u^P - \partial_y^2 u^P = 0 \\ \partial_x u^P + \partial_y v^P = 0 \\ u^P|_{y=0} = v^P|_{y=0} = 0, \quad \lim_{y \to +\infty} u^P(t,x,y) = \bar{u}(t,x). \end{cases}$$

$$v^P = -\int_0^y \partial_x u^P(z)\,dz.$$

$$\Rightarrow \begin{cases} \partial_t u^P + u^P \partial_x u^P - \left(\int_0^y \partial_x u^P(z)\,dz\right) \partial_y u^P - \partial_y^2 u^P = 0 \\ u^P|_{y=0} = 0, \quad \lim_{y \to +\infty} u^P(t,x,y) = \bar{u}(t,x). \end{cases}$$

Feature: Loss of derivative