Representations of complex semisimple Lie algebras I: Category O

Kei Yuen Chan

Shanghai Center for Mathematical Science, Fudan University

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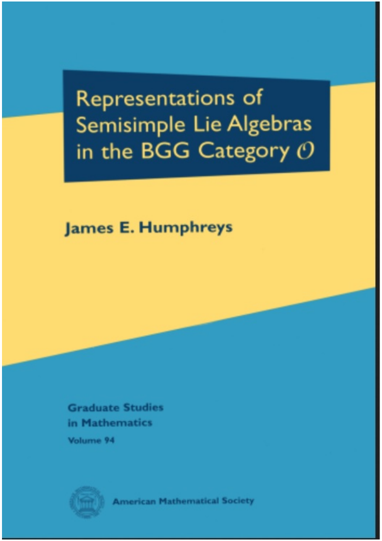
Reference


Representations of
Semisimple Lie Algebras
in the BGG Category $\mathcal{O}$
James E. Humphreys
Graduate Studies
in Mathematics
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Outline

- Basic notions for semisimple Lie algebras
- Category O: properties and modules
- Some homological properties

I. Basic notions for semisimple Lie algebras

Complex semisimple Lie algebras

- $\mathfrak{g}$: complex semsimple Lie algebras
- There is a 1-1 correspondence:

isom. classes of cplx simple Lie alg $\leftrightarrow$ connected root systems

- Fix a Cartan subalgebra $\mathfrak{h}$: a maximal toral in $\mathfrak{g}$
- Roots are given by eigenfunctions $\alpha : \mathfrak{h} \to \mathbb{C}$: for some $x \in \mathfrak{g} \setminus \{0\}$,

$$h.x = \alpha(h)x \quad \forall h \in \mathfrak{h}$$

- Cartan decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : h.x = \alpha(h)x \quad \forall x \in \mathfrak{h}\}.$$

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dim = 1

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Root systems and Weyl groups

- Fix a Borel subalgebra $\mathfrak{b}$, equivalently fix a set Φ^+ of positive roots
- Φ^+ determines a set Δ of simple roots
- A reflection s_α associated a root α satisfies:

$$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

  (for some inner form $\langle , \rangle$)
- Embed reflections into $\mathrm{O}(V)$, where $V = \mathbb{R} \otimes_{\mathbb{Z}} \Phi$
- W is the Weyl group generated by the reflections in $\mathrm{O}(V)$
- Example: $S_2 = \{1, s\}$,

$$S_3 = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$$

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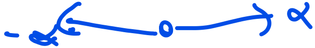

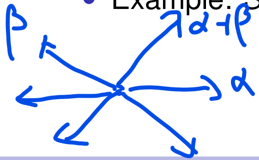

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α β $s_{\alpha+\beta}$

Universal enveloping algebra 张

- Recall that the tensor algebra $T(A)$ is defined as: $\bigoplus_{i=0}^{\infty} A^{\otimes i}$ with product $A^{\otimes i} \times A^{\otimes j} \to A^{\otimes (i+j)}$

$$(x_1 \otimes \ldots \otimes x_i)(y_1 \otimes \ldots \otimes y_j) = (x_1 \otimes \ldots \otimes x_i \otimes y_1 \otimes \ldots \otimes y_j)$$

- The universal enveloping algebra $U(\mathfrak{g})$ is defined as:

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g} \rangle$$

- Triangular decomposition: $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}$, where

$$\mathfrak{u}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

- PBW basis theorem: $U(\mathfrak{g})$ admits a basis:

$$\left\{ Y_{\alpha_1}^{r_1} \ldots Y_{\alpha_m}^{r_m} H_1^{s_1} \ldots H_l^{s_l} X_{\alpha_1}^{t_1} \ldots X_{\alpha_m}^{t_m} \right\}_{r_i, s_j, t_k \in \mathbb{Z}_{\geq 0}},$$

where $l = \mathrm{rank}\, \mathfrak{h}$ and $\{\alpha_i\}_{i=1}^{m}$ are all roots.

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$A = A^{\otimes 1}$

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$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook] \arrow[d, "\varphi"'] & U(\mathfrak{g}) \arrow[dl, dotted, "\psi"] \\ A & \end{tikzcd}$$

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$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$

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$\mathfrak{g}$ is abelian $\Rightarrow$ $U(\mathfrak{g}) \cong$ symmetric algebra of $\mathfrak{g}$

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$Y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$

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$H_1, \ldots, H_l$ any basis of $\mathfrak{h}$

where $l = \operatorname{rank} \mathfrak{h}$ and $\{\alpha_i\}_{i=1}^m$ are all positive roots.

Weight space

- $\mathfrak{h}^*$: linear functionals on $\mathfrak{h}$
- For $\lambda \in \mathfrak{h}^*$ and a $\mathfrak{g}$-module M, define

$$M_\lambda = \{x \in M : h.x = \lambda(h)x\}.$$

- In general, M_λ is not necessarily semisimple.
- For example, $\mathfrak{h} = \mathbb{C}.h$ and

$$h = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \quad \text{act on } M = \mathbb{C}^2$$

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weight space $$M_\lambda = \{x \in M : h.x = \lambda(h)x\}\,.$$

when $M_\lambda \neq 0$, M_λ is weight space & λ is a weight

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$\dim \mathfrak{h} = 1$

$M \neq \mathbb{C} \oplus \mathbb{C}$ as $\mathfrak{h}$-mod.

II. Categroy O: properties and modules

✓✓

- Correspondence $U(\mathfrak{g})$-modules and $\mathfrak{g}$-modules (via the embedding $\mathfrak{g} \mapsto U(\mathfrak{g})$)

$U(\mathfrak{g})$-mod M

i

$\mathfrak{g}$

$X.m = i(X)m$

Definition

Category O is defined as the full subcategory of $U(\mathfrak{g})$-modules such that

- M is finitely-generated $U(\mathfrak{g})$-modules
- $\mathfrak{h}$ acts semisimply i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$
- M is locally $\mathfrak{n}$-finite

The most basic examples of objects of Category O are finite-dimensional modules. (First and third bullets are easier, and we shall see the second one from Verma modules)

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- M is locally $\mathfrak{n}$-finite for any $v \in M$, $\dim \mathfrak{n}.v < \infty$

$\mathfrak{n} = \mathfrak{n}$

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no req. for $\mathfrak{n}^-$

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$\mathcal{O}_{\mathfrak{h},\mathfrak{b}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{h}',\mathfrak{b}'}$

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The most basic examples of objects of Category O are finite-dimensional modules. (First and third bullets are easier, and we shall see the second one from Verma modules)

Category $\mathcal{O}$ $\Rightarrow$ locally $U(\mathfrak{n})$-finite

Further properties of Category O

$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$

Proposition

Any module M in Category O satisfies:

- All weight spaces of M are finite-dimensional

axiom

axiom

- Let
$$\Lambda = \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\alpha.$$
There exists a finite set $\{\lambda_1, \ldots, \lambda_k\}$ of weights such that any weight in M takes the form:
$$\lambda_i - \lambda$$
for $\lambda \in \Lambda$.
- For any $M \in O$ and a finite-dimensional $L \in O$, $M \otimes L \in O$.
- M is a finitely-generated $U(\mathfrak{n}^-)$-module.
- For any $v \in M$, $Z(\mathfrak{g}).v$ is finite-dimensional.

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the action
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Rmk: not true if L is not finite-dim.

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implies

Center of $U(\mathfrak{g})$

Further properties of Category O

Proposition

- Category O is Noetherian.
- Category O is abelian.
- Category O is Artinian i.e. any module is of finite length

- Noetherian: Follows from $U(\mathfrak{g})$ is noetherian ring, and first axiom of Cat O
- Abelian: Check quotients and submodules. e.g. Submodules are $U(\mathfrak{g})$-finitely generated, and last two axioms easier
- Artinian: Need a bit more, and discuss alter

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Recall that $\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$. It has a basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying commutation relations:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

For $n \in \mathbb{Z}_{\geq 0}$, define M to be a module of basis $\{v_i\}_{i \geq 0}$ with action:

$$H.v_i = (n - 2i)v_i$$

$$X.v_i = (n - i + 1)v_{i-1}$$

$$Y.v_i = (i + 1)v_{i+1}$$

$\boxed{n=0}$

$v_3 \quad v_2 \quad v_1 \quad v_0 \quad 0$

$X \quad X \quad X \quad X$

H-action: $-6 \quad -4 \quad -2 \quad 0$

Modules which is not in Category O

locally

- $U(\mathfrak{g})$ as $U(\mathfrak{g})$-module (not $\mathfrak{n}$-finite)
- $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}$ (not $\mathfrak{h}$-semisimple)
- Product of two infinite-dimensional modules in O

$\mathbb{C}$ - trivial $U(\mathfrak{n})$ module

$U(\mathfrak{g})$ acts on left for the copy $U(\mathfrak{g})$ in $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}$

$u1 \rightsquigarrow u \otimes 1$

Highest weight modules

We now introduce an important class of modules in Cat O.

Definition

A $\mathfrak{g}$-module M is said to be of highest weight if

- there exists $v \in M$ such that $\mathfrak{n}.v = 0$;
- $M = U(\mathfrak{g}).v$.

We shall call v to be a maximal vector and the corresponding weight to be the highest weight.

Let M be a highest weight module with highest weight λ.

- Using PBW basis, any weight takes the form $\lambda - \sum n_\alpha \alpha$, where $n_\alpha \geq 0$.
- The highest weight and maximal vector are unique.
- A highest weight module is indecomposable.

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We shall call v to be a maximal vector and the corresponding weight to be the highest weight.

Let M be a highest weight module with highest weight λ.

- Using PBW basis, any weight takes the form $\lambda - \sum n_\alpha \alpha$, where $n_\alpha \geq 0$.
- The highest weight and maximal vector are unique.
- A highest weight module is indecomposable.

Highest weight modules

We now introduce an important class of modules in Cat O.

Definition

A $\mathfrak{g}$-module M is said to be of highest weight if

push down weights

a weight vector

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- $M = U(\mathfrak{g}).v$. $= U(\mathfrak{n}^-).v$ $\quad U(\mathfrak{g}) = \boxed{U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})} \otimes v$

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$U(\mathfrak{n}).v = \mathbb{C}v$

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Verma modules

Fix a weight $\lambda \in \mathfrak{h}^*$. Let v_λ be the λ-weight vector of $\mathfrak{h}$ and extend to $\mathfrak{b}$-module via $\mathfrak{n}$ acting trivially. Define Verma module to be

$\mathfrak{g}$ acts on left

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\lambda .$$

$Y.(X \otimes 1) = YX \otimes 1$

Indeed, for any highest weight module M, we find a surjection:

$$M(\lambda) \to M$$

determined by sending v_λ to the highest weight.

Proposition

Any Verma module has a unique quotient, denoted $L(\lambda)$.

Proof.

Sum of proper submodules does not contain the max. vector. □

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$M(\lambda)$ generates all highest weight modules

$$M(\lambda) \to M = U(\mathfrak{n}^-) v_\lambda' \leftarrow \text{max vector}$$

$1 \otimes v_\lambda \longmapsto v_\lambda'$ ⇒ extend to well-defined map

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Artinian property

Recall that we have the following:

Theorem

Category O is Artinian.

The idea of the proof is as follows:

- Locally $\mathfrak{n}$-finite property $\Rightarrow$ surjections of finite sum of Verma mods to a module in O
- Reduces to check finite length for Verma mdoules
- Any two simple modules $L(\lambda_1)$ and $L(\lambda_2)$ have non-trivial extension only if λ_1 and λ_2 are linked i.e.

$$\lambda_1 = w \cdot \lambda_2 := w(\lambda_2 + \rho) - \rho$$

  $(\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha)$
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$L(-2\rho)$ $L(0)$

-2ρ 0

$s(\rho)-\rho$

$0 \to L(\lambda_1) \to L'' \to L(\lambda_2) \to 0$

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finite dim on highest weight

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Simple module

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Duals in Category O

- There is an anit-involution $\tau : \mathfrak{g} \to \mathfrak{g}$ such that

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- For $M \in O$, define $M^\vee$ as:

$$M^\vee = \oplus_\lambda M_\lambda^\vee$$

with action given by:

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dual the space

Naive: $f \in M^*$

$(x.f)(v) := f(x.v)$

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$\{f : M_\lambda \to \mathbb{C}\}.$

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III. Homological properties

Some homological properties of Category O

- The ordering on weights is given by:

$$\lambda \leq \mu \quad \Leftrightarrow \quad \mu - \lambda \in \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\alpha$$

- If $\lambda \leq \mu$, then

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Projective objects

- Recall that an object P is said to be projective if the functor $\mathrm{Hom}_{O}(P, .)$ is right exact.
- For each simple object $L(\lambda)$, $\exists$ a projective cover $P(\lambda)$ for $L(\lambda)$ i.e. surjection from $P(\lambda)$ to $L(\lambda)$ and there is no proper projective submodule of $P(\lambda)$ mapping onto $L(\lambda)$
- Any projective object is a finite sum of projective covers
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Examples of projective objects

- In $\mathfrak{sl}(2,\mathbb{C})$ case, we consider simple modules with the 'infinitesimal character' 0: $L(0)$ and $L(-\rho)$
- The corresponding projective covers have structure:

$$P(0): \begin{array}{c} L(0) \\ L(-\rho) \end{array} \qquad P(-\rho): \begin{array}{c} L(-\rho) \\ L(0) \\ L(-\rho) \end{array}$$

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Construction of projective objects

- If λ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$, then $M(\lambda)$ is projective.
- Now for arbitrary weight λ, we find a sufficiently large n such that $\mu = \lambda + n\rho$ is dominant.
- The module $M(\mu) \otimes L(n\rho)$ is projective.
- There is a general theory to decompose $M(\mu) \otimes L(n\rho)$. In particular, $M(\lambda)$ appears once.
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BGG reciprocity

One important result that relates projective objects, simple objects and Verma modules is the BGG reciprocity:

- By previous construction, projective object admits a filtration with successive subquotients are isomorphic to Verma mdoules. We call *standard filtration*.
- The number of times that $M(\mu)$ appears in the filtration of $P(\lambda)$ is independent of the choice of the filtration.
- Denote the number by $(P(\lambda) : M(\mu))$.

Theorem (BGG reciprocity)

Let $\lambda, \mu \in \mathfrak{h}^*$. *Then*

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$$

Key of proof: $(P(\lambda) : M(\mu)) = \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^{\vee})$

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Projective modules: antidominant case

Following from the projective object $M(-\rho) \otimes M(\lambda + \rho)$, we have:

Proposition

Let λ be an antidominant weight. Then

$$(P(\lambda) : M(\mu)) = 1$$

for any μ with $w \cdot \mu = \lambda$ for some $w \in W$.

As a consequence of BGG reciprocity, we als have that

$$[M(\mu) : L(\lambda)] = 1.$$

Indeed, $L(\lambda)$ appears in the submodule $M(\mu)$. We will come back in the last lecture.