Representations of complex semisimple Lie algebras II: Kazhdan-Lusztig theory

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Outline

- Character problem
- Translation functor (reducing problem to a single case)
- Kazhdan-Lusztig theory

I. Character problem

Central characters

- $Z(\mathfrak{g})$: center of $U(\mathfrak{g})$
- Harish-Chandra isomorphism:

$$Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$$

given by: write z in PBW basis and then take the part of the form $H_1^{m_1} \dots H_r^{m_r}$, where $H_1, \dots, H_r$ forms a basis for $\mathfrak{h}$

- Schur's lemma: $Z(\mathfrak{g})$ acts by a scalar function χ_λ on $L(\lambda)$
- It turns out that $\chi_\lambda = \chi_\mu$ iff $\lambda = w \cdot \mu := w(\mu + \rho) - \rho$ for some $w \in W$
- Block decomposition:

$$O = \bigoplus_\lambda O_{\chi_\lambda}$$

Remark: O_{χ_λ} is not indecomposable in general

- We shall assume all λ is integral i.e. $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all coroots $\alpha^\vee$

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- For $M \in O$ and $\lambda \in \mathfrak{h}^*$, recall that

$$M_\lambda = \{v \in M : h.v = \lambda(h)v \quad \text{for all } h \in \mathfrak{h}\}.$$

- The formal character of M is defined as:

$$\mathrm{ch}\, M = \sum_\lambda e^\lambda \dim M_\lambda$$

- Multiplication: $e^{\lambda_1} * e^{\lambda_2} = e^{\lambda_1 + \lambda_2}$
- Tensor product $M_1 \otimes M_2$ is defined as:

$$X.(v_1 \otimes v_2) = (X.v_1) \otimes v_2 + v_1 \otimes (X.v_2)$$

  and $\mathrm{ch} M_1 \otimes M_2 = \mathrm{ch} M_1 * \mathrm{ch} M_2$
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- Define the partition function $p(\lambda)$ to be the number of tuples $(n_\alpha)_{\alpha\in\Lambda}$ in $\mathbb{Z}_{\geq 0}$ such that $\lambda = \sum_\alpha n_\alpha \alpha$.
- Define $p = \sum_\lambda p(\lambda)e^{-\lambda}$
- Following from PBW basis, as $\mathfrak{h}$-modules

$$M(\mu) = U(\mathfrak{n}^-) \otimes \mathbb{C}v_\mu$$

The weights from $U(\mathfrak{n}^-)$ is determined by p and so

$$\mathrm{ch}\, M(\mu) = p * e^\mu$$

- To work for $\mathrm{ch}(L(\mu))$, one has to write a Verma module $[M(\lambda)]$ as a linear combination of simple modules $[L(\mu)]$.

Formal characters of Verma modules

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Verma modules and simple modules

- On the other hand, one may also do the converse:

$$[M(\lambda)] = \sum_{\mu} a_{\lambda,\mu}[L(\mu)]$$

- The problem is to determine $a_{\lambda,\mu}$
- We could first reduce to:

$$[M(\lambda)] = \sum_{w} a_w[L(w \cdot \lambda)],$$

where $a_w = a_{\lambda, w\cdot\lambda}$.
- We have that $a_{\lambda,\lambda} = 1$ and $a_{\lambda,w\cdot\lambda} = 0$ for $w \cdot \lambda \not\leq \lambda$.
- We shall consider the 'inverse problem':

$$[L(\lambda)] = \sum_{w} b_w[M(w \cdot \lambda)].$$

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II. Translation functor (reducing problem to a single case)

Translation functor

To compute those $a_{\lambda,\mu}$, our goal is to reduce to a particular central character. In order to do so, we need a translation functor.

- Translation functor $T_\lambda^\mu : O_{\chi_\lambda} \to O_{\chi_\mu}$ is defined as

$$\mathrm{pr}_\mu(L \otimes \mathrm{pr}_\lambda(M)),$$

  where
  - $\mathrm{pr}_\lambda : O \to O_{\chi_\lambda}$;
  - $L = L(\bar{\nu})$ is the finite-dim module with central character $\chi_{\mu-\lambda}$
- It is exact and satisfies:

$$T_\mu^\lambda M^\vee \cong (T_\mu^\lambda M)^\vee$$

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Facet

- A facet F is a subspace of $\mathfrak{h}^*$ determined by a tri-partition $\Phi_F^+, \Phi_F^0, \Phi_F^-$ of Δ satisfying the following

$$\lambda \in F \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \Phi_F^+ \\ \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \Phi_F^0 \\ \langle \lambda + \rho, \alpha^\vee \rangle < 0 & \text{when } \alpha \in \Phi_F^- \end{cases}$$

- For such λ, we shall call F to be the facet of λ.

Translation functor on Verma modules

- We say that λ is anti-dominant if $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ for all coroots $\alpha^\vee$.

Assume λ and μ are anti-dominant. Suppose λ lies in a facet F and μ lies in $\overline{F}$.

- $T_\lambda^\mu M(w \cdot \lambda) = M(w \cdot \mu)$
- Idea of proof:

$$M(w \cdot \lambda) \otimes L(\bar{\nu}) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{w \cdot \lambda}) \otimes L(\bar{\nu}) \quad (1)$$

$$\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{w \cdot \lambda} \otimes L(\bar{\nu})) \quad (2)$$

- Categorical equivalence: If λ and μ lie in the same facet, then T_λ^μ and T_μ^λ define an equivalence of categories.

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Translation functor on simple modules

- Translation functor on simple modules:

$$T_\lambda^\mu L(w\cdot\lambda) = L(w\cdot\mu) \quad \text{or} \quad 0$$

- To determine when it is zero or not, we need a notion of the upper closure:

$$\lambda \in \widehat{F} \Leftrightarrow \begin{cases} \langle \lambda+\rho, \alpha^\vee\rangle > 0 & \text{for } \alpha \in \Phi_F^+ \\ \langle \lambda+\rho, \alpha^\vee\rangle = 0 & \text{for } \alpha \in \Phi_F^0 \\ \langle \lambda+\rho, \alpha^\vee\rangle \le 0 & \text{for } \alpha \in \Phi_F^- \end{cases}$$

- In $\mathfrak{sl}(2,\mathbb{C})$ case, if $\Phi_F^+ = \{\alpha\}$ or $\Phi_F^0 = \{\alpha\}$, then $\widehat{F} = F$.
- In $\mathfrak{sl}(2,\mathbb{C})$ case, if $\Phi_F^- = \{\alpha\}$, then the upper closure contains the half line $\le -\rho$.

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Translation functor on simple modules

- Suppose λ lies in a facet F and $w \cdot \mu$ lies in $\widehat{w \cdot F}$. Then
$$T_\lambda^\mu L(w \cdot \lambda) \cong L(w \cdot \mu)$$
- Suppose λ lies in a facet F and $w \cdot \mu$ does not lie in $\widehat{w \cdot F}$. Then
$$T_\lambda^\mu L(w \cdot \lambda) = 0$$

So, now if we have a character formula

$$[L(w \cdot \lambda)] = \sum_{w' \cdot \lambda \leq w \cdot \lambda} a_{w'} [M(w' \cdot \lambda)],$$

Transferring problem

- Note that -2ρ is regular and let F^* be the facet of -2ρ. We have that
$$V = \sqcup_{w \in W} \widehat{w \cdot F^*}$$
- Change w if necessary, we can find $w \cdot \mu$ lies on $\widehat{w \cdot F^*}$. Hence, we can always transfer the problem of determining
$$[L(w \cdot \mu)] = \sum_{w' \cdot \lambda \leq w \cdot \lambda} a_{w'} [M(w' \cdot \mu)].$$
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Rephrasing the problem

Let $\lambda = -2\rho$. Let $M_w = M(w \cdot \lambda)$ and $L_w = L(w \cdot \lambda)$. We have to determine the coefficients $a_{x,w}$ in

$$[L(w)] = \sum_{x \leq w} a_{x,w}[M_x]$$

To this, we need to introduce another technique– Hecke algebra.

Example

In $\mathfrak{sl}(2,\mathbb{C})$,

$$[L(0)] = [M(0)] - [M(-2\rho)]$$

In $\mathfrak{sl}(3,\mathbb{C})$,

$$[L(-2(\alpha+\beta))] = [M(-2(\alpha+\beta))]$$

$$[L(-2\alpha-\beta)] = [M(-2\alpha-\beta)] - [L(-2(\alpha+\beta))]$$

$$[L(-\alpha-2\beta)] = [M(-\alpha-2\beta)] - [L(-2(\alpha+\beta))]$$

$$[L(-\beta)] = [M(-\beta)] - [M(-2\alpha-\beta)]$$

$$[L(-\alpha)] = [M(-\alpha)] - [M(-\alpha-2\beta)]$$

$$[L(0)] = [M(0)] - [M(-\alpha)] - [M(-\beta)] + [M(-\alpha-2\beta)] + [M(-2\alpha-\beta)$$
$$- [M(-2\alpha-2\beta)]$$

III. Kazhdan-Lusztig theory

Hecke algebra

Definition

Let $A = \mathbb{Z}[q, q^{-1}]$ be the Laurent polynomial ring. Let W be the Weyl group. The Hecke algebra $\mathcal{H}_{W,q}$ is the associative algebra with generators $\{T_w\}_{w \in W}$ with relations:

- (Braid relation) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$
- (Quadratic relation) $T_s^2 = (q-1)T_s + q$ for simple reflection s.

We identify T_1 with the identity.

- When specializing to $q = 1$, $\mathcal{H}_{W,1}$ is isomorphic to $\mathbb{C}[W]$.
- (PBW basis thm) $\{T_w\}_{w \in W}$ forms a PBW basis.

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Aside on Hecke algebra

The Hecke algebra above is the one we need. In general, one may consider a more general algebra with generators $\{T_w\}_{w \in W}$ with relations:

- $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$
- $(T_s + 1)(T_s - q_s) = 0$ for simple reflections s and some q_s

Note that q_s is possibly distinct here. A PBW basis exists if and only if $q_s = q_{s'}$ if s and s' are in the same W-conjugacy class.

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Aside on Hecke algebra

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Involution on Hecke algebra

- Define an involution $\iota : \mathcal{H} \to \mathcal{H}$ by $\iota(T_w) = T_{w^{-1}}^{-1}$
- To check it is well-defined, one has to check ι preserves the relations:
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$$T_{(w_1 w_2)^{-1}} = T_{w_2^{-1}} T_{w_1^{-1}} \Rightarrow \iota(T_{w_1 w_2}) = \iota(T_{w_1})\iota(T_{w_2})$$

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- For $p = \sum_n a_n q^n$, define $p^* = \sum_n a_n q^{-n}$, and extends to $\mathcal{H}$
- Define $\bar{\_} : \mathcal{H} \to \mathcal{H}$ as: $\overline{h} = \iota(h^*)$

Theorem (Kazhdan-Lusztig)

There exists unique elements $C_w \in \mathcal{H}$ $(w \in W)$ such that

- $\overline{C_w} = C_w$
- $$C_w = (-1)^{l(w)} q^{l(w)/2} \sum_{x \leq w} (-1)^{l(x)} q^{-l(x)} \overline{P_{x,w}(q)} T_x,$$

  where

  - $P_{w,w}(q) = 1$,
  - $P_{x,w}(q) \in \mathbb{Z}[q]$,
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Some examples

Example

- $C_1 = T_1$
- $C_s = q^{-1/2}(T_s - q) = -q^{1/2}(-q^{-1}T_s + 1)$. Hence, we have that $P_{1,s} = 1$.

Kazhdna-Lusztig conjecture (theorem)

Theorem

Recall that $\lambda = -2\rho$. Define $L_w = L(w \cdot \lambda)$ and $M_w = M(w \cdot \lambda)$. Then

$$[L_w] = \sum_{x \leq w} (-1)^{l(w)-l(x)} P_{x,w}(1)[M_x].$$

Of course, a subtly at the first glance is that Hecke algebra has nothing to do with Category O.

Connection to geometry

A connection to geometry is the following result. Let G be the complex algebraic group over $\mathbb{C}$ and let B be the Borel subgroup. The Schubert variety for G is G/B. For $w \in W$, define $X_w = BwB/B$, which is a locally closed submanifold of X. We denote $\mathrm{IC}(Y)$ by the intersection complex of Y, which is for singular variety.

Theorem (Kazhdan-Lusztig)

For $x \leq w$ in W,

$$P_{x,w}(q) = \sum_{i=0}^{l(w)} q^i \dim H^{2i}(\mathrm{IC}(X_w))_x$$

One further needs Riemann-Hilbert correspondence to link to D-modules, and then to $U(\mathfrak{g})$-modules via Beilinson-Bernstein correspondence.

Homological formulation

Theorem (Vogan)

For $x \leq w$,

$$P_{x,w}(q) = \sum_{i \geq 0} q^i \dim \mathrm{Ext}_O^{l(w)-l(x)-i}(M_x, L_w)$$

Example

In $\mathfrak{sl}(2, \mathbb{C})$, L_s is the trivial representation, and $\dim \mathrm{Ext}_O^1(M_1, L_s) = 1$. Indeed, the extension corresponds to the dual of the Verma module M_s.

Thank you