Representations of complex semisimple Lie algebras III: Jantzen conjecture

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Kazhdan-Lusztig polys

Hecke alg • $\{T_w\}_{w \in W}$

higher -representation

• $\{C_w\}_{w \in W}$ • $\overline{C_w} = C_w$

• $C_w = (-1)^{l_w} \sum \ldots P_{x,w}(q)$

- Jantzen filtration
- Embedding problem
- Jantzen conjecture

I. Jantzen filtration

Jantzen filtration

Recall in the last lecture, we have the Kazhdan-Lusztig conjecture:

Theorem

In the block O_0 (with infinitesimal character as -2ρ), for $x \leq w$ in W,

$$[L_w] = \sum_{x \leq w} (-1)^{l(w)-l(x)} P_{x,w}(1)[M_x]$$

Here $P_{x,w}(q)$ is so-called Kazhdan-Lusztig polynomial. But, can one have interpretation on the variable q?

- Recall that we have a transpose map τ of $U(\mathfrak{g})$ with properties:

$$\tau(\mathfrak{h}), \quad \tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}, \quad \tau(\mathfrak{g}_{-\alpha}) = \mathfrak{g}_\alpha$$

e.g. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ matrix alg. $\tau(X) = X^T$

- A symmetric bilinear form $(.,.)$ on M is called contravariant if

$$(u.m_1, m_2) = (m_1, \tau(u).m_2)$$

- $\mathrm{Bil}(M) \cong \mathrm{Hom}_{\mathcal{O}}(M, M^\vee)$
- In particular, there is a canonical contravariant form on $M(\lambda)$, which is non-degenerate only if $M(\lambda)$ is irreducible. Indeed, the kernel of such form is the maximal submodule of $M(\lambda)$.

Contravariant form

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anti-involut.

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$$(u.m_1, m_2) = (m_1, \tau(u).m_2)$$

$$(X_1 X_2 m_1, m_2) = (m_1, \tau(X_2)\tau(X_1).m_2).$$

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$$M^\vee$$

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$(\,,\,) \mapsto (m \mapsto (m, -))$ define the dual using τ.

- $\mathrm{Bil}(M) \cong \mathrm{Hom}_{\mathcal{O}}(M, M^\vee)$ $\mathrm{Hom}_{\mathcal{O}}(M(\lambda), M(\lambda)^\vee)$ $\to \mathbb{C}[\lambda]$
- In particular, there is a canonical contravariant form on $M(\lambda)$, which is non-degenerate only if $M(\lambda)$ is irreducible. Indeed, the kernel of such form is the maximal submodule of $M(\lambda)$. $\forall\, m \in M,\ (m, m') \neq 0$ for some m'

The $\mathfrak{sl}(2,\mathbb{C})$ example

$\{H, X, Y\}$ $[H,X]=2X$, $[H,Y]=-2Y$
$[X,Y]=H$

For $n \in \mathbb{Z}_{\geq 0}$, define M to be a module of basis $\{v_i\}_{i\geq 0}$ with action:

$$H.v_i = (n + T - 2i)v_i$$

$$X.v_i = (n + T - i + 1)v_{i-1}$$

$$Y.v_i = (i+1)v_{i+1}$$

T variable

$\mathrm{Hom}(M, M^\vee) \cong \mathbb{C}$

Set $\boxed{(v_0, v_0) = 1}$. We can compute (v_i, v_i) as follows:

$$(i+1)(v_{i+1}, v_{i+1}) = (Y.v_i, v_{i+1}) = (v_i, X.v_{i+1}) = (n+T-i)(v_i, v_i)$$

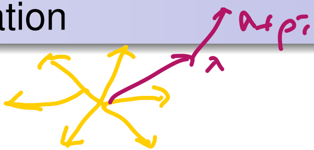
Thus, $(v_i, v_i) = \frac{(n+T+1-i)\ldots(n+T+1-0)}{i!}(v_0, v_0)$. When $i \geq n+1$,

$$T \mid (v_i, v_i).$$

divides

$(v_i, v_i) = T\,P(T)$
$P(0) \neq 0$

Jantzen filtration



- Let T be a variable and let $\mathfrak{g}_T = \mathbb{C}[T] \otimes_{\mathbb{C}} \mathfrak{g}$. Let $A = \mathbb{C}[T]$. Define $\lambda_T = \lambda + \rho T$.
- So, for generic T, we have that $M(\lambda_T)$ is irreducible. Hence $\exists$ non-degenerate contravariant form $(,)$ on $M(\lambda_T)$.
- We define

$$M(i) = \left\{ e \in M(\lambda_T) : (m, M(\lambda_T)) \in T^i M(\lambda_T) \right\}.$$

- Note that $M(\lambda) \cong \overline{M} := M(\lambda_T)/TM(\lambda_T)$. Define M^i to be the image of $M(i)$ in $\overline{M}$.
- In previous example, $M^0 = M(\lambda)$ and $M^1 = \text{max sub mod}$

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- We define $i \geq 0$,

$T^i A \quad T^i (1, T, \ldots)$

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Let $T = 0$

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Jantzen filtration

Theorem (Jantzen)

The above filtration ↑ previous page

$$\overline{M} = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \dots$$

gives that

- *each non-zero subquotient $M(\lambda)^i/M(\lambda)^{i+1}$ has a non-degenerate contravariant form* ← comes from the one
- *M^1 is the maximal proper submodule of $M(\lambda)$* of $M(\lambda_T)$

There is one more property called Jantzen character sum formula, which we will state later.

Jantzen filtration

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- Radical filtration:

$$\mathrm{Rad}^0 M = M \supset \mathrm{Rad}^1 M \supset \mathrm{Rad}^2 M \supset \dots$$

such that $\mathrm{Rad}^i M/\mathrm{Rad}^{i+1} M$ is maximal semisimple quotient of $\mathrm{Rad}^i M$.

- Socle filtration:

$$\mathrm{Soc}^0 M = 0 \subset \mathrm{Soc}^1 M \subset \mathrm{Soc}^2 M \subset \dots$$

such that $\mathrm{Soc}^i M/\mathrm{Soc}^{i-1} M$ is maximal semisimple submodule of $M/\mathrm{Soc}^{i-1} M$.

Radical and socle filtration

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One may ask if radical filtration and socle filtration coincides in general.

Example

Radical = $\begin{array}{c} A \oplus C \\ B \end{array}$

Module:

$$\begin{array}{c} A \\ B \end{array} \oplus C$$

Socle = $B \oplus C \to \begin{array}{c} A \\ B \oplus C \end{array}$

where A, B, C are simple. Then Rad $\neq$ Soc.

Let M be a module with unique simple submodule and unique simple quotient. Then the socle filtration of M agrees with the radical filtration of M (with suitable relabelling). We shall call such M to be rigid.

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Submodule of Verma modules

We have already known that a Verma module has a unique quotient. To show a Verma module is rigid, it remains to show that a Verma module has a unique submodule.

Proposition

- Suppose λ is antidominant. Then $M(\lambda)$ is irreducible i.e. $M(\lambda) = L(\lambda)$.
- Any Verma module $M(\lambda)$ has a unique submodule. The unique submodule is isomorphic to $M(w \cdot \lambda)$ with $w \cdot \lambda$ antidominant.

To prove uniqueness, one realize $M(\lambda)$ as $U(\mathfrak{n})$, as $\mathfrak{n}$-modules. Any two non-zero ideals of $U(\mathfrak{n})$ have non-zero intersection.

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Proposition $\langle \lambda+\rho, \alpha^\vee \rangle \leq 0$ for all $\alpha^\vee$

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PBW

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$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$

Jantzen conjecture

Theorem

- *Each Verma module is rigid.*
- *The Lowey length of M_w is $l(w) + 1$.*
- *The Jantzen filtration coincides with the radical filtration. In particular, each Jantzen layer is semisimple.*
- *The radical filtration is determined by the Kazhdan-Lusztig polynomails:*

$$P_{w_0 w, w_0 x}(q) = \sum_k [\mathrm{Rad}_{l(x,w)-k} M_w, L_w] q^k$$

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II. Embedding problem

Some submodule structure

There are some related problems to above version of Jantzen conjecture. We have mentioned that any Verma module has a unique submodule. More generally, one may ask:

$$\dim \mathrm{Hom}_{\mathcal{O}}(M(\lambda), M(\mu)) =?$$

Indeed, uniqueness of submodule determines:

- If there is a non-zero map from $M(\lambda)$ to $M(\mu)$, then the map is an embedding.
- $\dim \mathrm{Hom}_{\mathcal{O}}(M(\lambda), M(\mu)) \leq 1$.

It remains to ask when

$$\dim \mathrm{Hom}_{\mathcal{O}}(M(\lambda), M(\mu)) \neq 0$$

It relies on a notion of 'linked'

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Strongly linked

For two weights $\lambda, \mu \in \mathfrak{h}^*$, $\mu \uparrow \lambda$ if

- $\lambda = \mu$ or
- there is a positive root α such that $\mu = s_\alpha \cdot \lambda \leq \lambda$.

Definition

We say that μ is strongly linked to λ if $\mu = \lambda$ or there exists positive roots $\alpha_1, \ldots, \alpha_r$ such that

$$\mu = (s_{\alpha_1} \ldots s_{\alpha_r}) \cdot \lambda \uparrow (s_{\alpha_2} \ldots s_{\alpha_r}) \cdot \lambda \uparrow \ldots \uparrow \lambda$$

Example

In $\mathfrak{sl}(2, \mathbb{C})$, for $k \geq 0$, $-(k+2)\rho \uparrow k\rho$. As we saw that $M(k\rho)$ has two composition factors $M(k\rho)$ and $M(-(k+2)\rho)$ and

$$M(-(k+2)\rho) \hookrightarrow M(k\rho)$$

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$L(k\rho)$

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BGG theorem

The previous observation can be generalized:

Theorem

- *(Verma) If μ is strongly linked to λ, then $M(\mu) \hookrightarrow M(\lambda)$. In particular, $[M(\lambda) : L(\mu)] \neq 0$.*
- *(BGG) If $[M(\lambda) : L(\mu)] \neq 0$, then μ is strongly linked to λ.*

- The idea of proving the first one is to reduce to the $\mathfrak{sl}_2$-calculation for each step

$$s_{\alpha_i} \cdot \lambda' \uparrow \lambda'$$

The second one needs some new ideas.

- Combine two parts: $[M(\lambda) : L(\mu)] \neq 0 \Leftrightarrow M(\mu) \hookrightarrow M(\lambda)$

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Jantzen sum formula

Recall that we have the Jantzen filtration:

$$M(\lambda) = M^0 \supset M^1 \supset M^2 \supset \dots$$

Theorem

The characters satisfy:

$$\sum_{i>0} \mathrm{ch}\, M^i = \boxed{\sum_{\alpha>0, s_\alpha \cdot \lambda < \lambda} \mathrm{ch}\, M(s_\alpha \cdot \lambda)}$$

Application on proving BGG reciprocity: Suppose $[M(\lambda) : L(\mu)] > 0$ and $\mu \neq \lambda$. Then $[M^1 : L(\mu)] \neq 0$. Above theorem implies $[M_{s_\alpha \cdot \lambda} : L(\mu)] \neq 0$. Induction implies μ strongly linked to $s_\alpha \cdot \lambda$, and so strongly linked to λ.

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$\mu \uparrow \dots \uparrow s_\alpha\cdot\lambda \uparrow \lambda$

More examples

Let us look at $\mathfrak{sl}(3, \mathbb{C})$ case. The Weyl group has order 6. The module structure $M(0)$ takes the form:

$L(0)$

$M(s_\alpha \cdot 0)$ → $L(s_\alpha \cdot 0)$ $\qquad$ $L(s_\beta \cdot 0)$ $\qquad$ $M(s_\beta \cdot 0)$

$L(s_\beta s_\alpha \cdot 0)$ $\qquad$ $L(s_\alpha s_\beta \cdot 0)$

$M(s_\beta s_\alpha \cdot 0)$ $\qquad$ $L(s_\alpha s_\beta s_\alpha \cdot 0)$

antidominant

III. More on Jantzen conjecture

Now we ask how Jantzen filtration behaves under the embedding of Verma modules. The following is the original Jantzen conjecture, which is a consequence of JF=RF

Theorem

Let $\mu \uparrow \lambda$ in $\mathfrak{h}^*$. Consider the embedding

$$M(\mu) \hookrightarrow M(\lambda).$$

Let $\Phi_\kappa^+ = \{\alpha > 0 : s_\alpha \cdot \kappa < \kappa\}$. Let $r = |\Phi_\lambda^+| - |\Phi_\mu^+|$. Then

$$M(\mu) \cap M(\lambda)^i = M(\mu)^{r-i}$$

for $i \geq r$. In particular, $M(\mu) \subset M(\lambda)^r$.

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$i-r$

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Flag variety

We discussed with connection with the geometry of flag variety G/B last time.

- Let $G = \mathrm{GL}_n(k)$, where $k = \mathbb{C}$ or $\overline{\mathbb{F}}_q$.
- Let B be the subgroup of upper triangular matrices. G/B can be identified with the space consisting of a sequence of linear subspaces of k^n

$$\{V_1 \subset V_2 \subset \ldots \subset V_n : \dim_k V_i = i\}$$

under the correspondence:

$$gB \leftrightarrow \{g\mathrm{span}(e_n) \subset g\mathrm{span}(e_n, e_{n-1}) \subset \ldots \subset g\mathrm{span}(e_n, \ldots, e_1)\}$$

- As a variety, G/B is projective.

Bruhat decomposition

- Recall that $W = N_G(T)/T$ is the Weyl group. Bruhat decomposition:
$$G = \sqcup_{w \in W} BwB$$
which gives a stratification on the B-orbits on G/B parametrized by W:
$$w \in W \leftrightarrow BwB/B$$

- For $G = \mathrm{GL}(2, \mathbb{C})$, G/B is $\mathbb{P}^1$. BsB/B corresponds to the open orbit $\{[1, y]\} \cong k$, and B/B corresponds to the point $\infty = [1, 0]$.
- For $G = \mathrm{GL}(3, \mathbb{C})$, G/B the has 6 B-orbits.
- The closure relation on G/B-orbits compatible with the Bruhat ordering on W

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Schubert cell,

B_sB

B

- For $G = \mathrm{GL}(2, \mathbb{C})$, G/B is $\mathbb{P}^1$. BsB/B corresponds to the open orbit $\{[1, y]\} \cong k$, and B/B corresponds to the point $\infty = [1, 0]$.
- For $G = \mathrm{GL}(3, \mathbb{C})$, G/B the has 6 B-orbits.
- The closure relation on G/B-orbits compatible with the Bruhat ordering on W

Little bit on D-modules

- For $G = \mathrm{SL}(2,\mathbb{C})$ case, $X = G/B = \mathbb{P}^1$. G acts by the transformation $[x:1]$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.x = \frac{ax+b}{cx+d}$$
- For functions on X, by taking differentiating one obtains corresponding $\mathfrak{g}$-action e.g. take $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then
$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \cdot z = \frac{z}{1+tz} \Rightarrow (e.\psi)(z) = -z^2 \frac{d}{dz}\psi$$
$$z = \frac{1}{x}$$
- This is how may construct modules from functions on X.
- One consider the 'sheaf version': that is D-modules, and taking globalization (roughly) gives equivalence corresp. on $U(\mathfrak{g})$-modules.

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Weight filtration on the geometry side

Let us briefly explain the idea of a proof of Jantzen filtration, due to Beilinson-Bernstein. Recall that to establish the KL conjecture, one needs:

global

- $\mathfrak{g}$-modules $\overset{\text{RH corr.}}{\longleftrightarrow}$
- D-modules on G/B $\overset{\text{BB corr.}}{\longleftrightarrow}$
- perverse sheaves on G/B ← other interpretation

The variety G/B is defined over $\overline{\mathbb{F}}_q$, and one has a Frobenius operator Fr acting on the variety. For a stalk of a sheaf of G/B, Fr acts by $(q^n)^{w/2}$. (n is fixed, but w depends on $\mathcal{F}_x$. w defines a graded, called weight. The perverse sheaves corresponding to Verma modules admit a weight filtration. BB shows it agrees with Jantzen fitlration.

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radical filtration

We basically consider complex semisimple Lie algebra $\mathfrak{g}$. How about if we study Lie groups G?
Like $\mathfrak{g}$, we first need a suitable class of representations of G.
Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

Definition

A $\mathfrak{g}$-module M is said to be a $(\mathfrak{g}, K)$-module if

- for any $m \in M$, $x \in \mathfrak{g}$, $k \in K$,
- $\frac{d}{dt}\exp(tX).m|_{t=0} = X.m$,
- (K-finiteness) for any $m \in M$, $K.m$ is finite-dimensional.

The definition seemingly comes from Lie algebra only, but there is a natural way to construct Lie group representation by completion of the underlying space.

Another generalization

One may a larger class of representations of $\mathfrak{g}$. Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be a parabolic subalgebra with Levi subalgebra $\mathfrak{l}$ and unipotent part $\mathfrak{u}$.

Definition

$O^{\mathfrak{p}}$ is a full subcategory of $\mathrm{Mod}\, U(\mathfrak{g})$ whose objects M satisfy:

- M is finitely-generated $U(\mathfrak{g})$-module;
- as $U(\mathfrak{l})$-module, M is sum of finite-dimensional $\mathfrak{l}$-modules;
- M is locally $\mathfrak{u}$-finite.